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ON THE DYNAMICS OF A VEHICLE ON PNEUMATIC TIRES

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An extension of Keldysh's theory [1] of rolling of a wheel with an elastic pneumatic tire to curvilinear motion at any speed along a trajectory of arbitrary curvature is proposed.

Hypotheses of rolling without slippage were formulated and appropriate equations of nonholonomic relations derived in [1] with the aim of establishing the conditions of aircraft landing gear shimmi excitation on the runway. Validity of that theory for curvilinear motion was questioned in [2]. On the assumption of validity of theory [1] for curvilinear motion the authors of [3] had to restrict their study to the motion along a "path of fairly small curvature at not very high speed".

A method based on the theory [1] and study [3] is proposed here for investigating the programmed motion along a trajectory of arbitrary curvature at any speed. The programmed motion is understood here to be the motion along a specified (programmed) curve of curvature $k = R^{-1}$ at the point K of intersection of the straight line drawn through the wheel center C (Fig. 1) in the median plane with the road plane.

We used the notation: O_1xyz for the inertial system of coordinates; O for the tire footprint center; x, y and x_*, y_* for the abscissas and ordinates of points K and O in the O_1xyz -system; $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{i}_1$, $\mathbf{j}_1$ for unit vectors of axes O_1x, O_1y and Kx_1, Ky_1; θ for heading; χ for the wheel angle of tilt produced by tire deformation; ξ for the linear lateral deformation of the tire; φ for the tire angular deformation (twist); Γ for the line of tire rolling; s and s_K for the lengths of arcs of tranjectories of point O and K, and $Ox'y'$ for the coordinate system the direction of whose axes differs from that of the O_1x and O_1y by the angle $\theta + \varphi$. We have the following relations:

$$\mathbf{v}_k = \mathbf{i}x^\cdot + \mathbf{j}y^\cdot = \mathbf{i}_1 v + \mathbf{i}_1 U, \quad v = s_K^\cdot = x^\cdot \cos\theta + y^\cdot \sin\theta \tag{1}$$

$$U = -x^\cdot \sin\theta + y^\cdot \cos\theta, \quad \mathbf{v}_O = \mathbf{i}x_*^\cdot + \mathbf{j}y_*^\cdot = \mathbf{i}'[x_*^\cdot \cos(\theta+\varphi) + y_*^\cdot \sin(\theta+\varphi)] + \mathbf{j}'[-x_*^\cdot \sin(\theta+\varphi) + y_*^\cdot \cos(\theta+\varphi)] = \mathbf{i}_1(v + \xi\theta^\cdot) + \mathbf{j}_1(U - \xi^\cdot)$$

$$s^\cdot = (s_K^\cdot + \xi\theta^\cdot)\cos\varphi + (U - \xi^\cdot)\sin\varphi, \quad x_* = x + \xi\sin\theta, \quad y_* = y - \xi\cos\theta$$

The first of Keldysh's hypotheses is expressed by the equation

$$-x_*^\cdot \sin(\theta+\varphi) + y_*^\cdot \cos(\theta+\varphi) = 0 \tag{2}$$

which according to (1) represents the condition $(\mathbf{v}_0)_{y'} = 0$ which for $\xi = 0$ and

$\varphi = 0$ becomes the usual equation $-x^{\cdot} \sin\theta + y^{\cdot} \cos\theta = 0$ of nonholonomic relationships for an absolutely rigid body which implies impossibility of lateral slipping.

Owing to tire deformation the trajectories of points O and K are different. We assume that the curvature of line Γ is an additive function of deformation parameters, and, in conformity with the linear theory of elasticity, set

$$d(\theta + \varphi) / ds = k + f(\xi, \varphi, \chi), \quad f(\xi, \varphi, \chi) = \alpha\xi - \beta\varphi - \gamma\chi$$

where α, β, γ are Keldysh's constants.

The equations of nonholonomic relationships linearized with respect to deformations are of the form

$$x^{\cdot} \sin\theta - y^{\cdot} \cos\theta + (x^{\cdot} \cos\theta + y^{\cdot} \sin\theta)\,\varphi + \xi^{\cdot} = 0 \tag{3}$$

$$\theta^{\cdot} + \varphi^{\cdot} - k\,[x^{\cdot} \cos\theta + y^{\cdot} \sin\theta + \varphi\,(-x^{\cdot} \sin\theta + y^{\cdot} \cos\theta) + \xi\theta^{\cdot}] - (x^{\cdot} \cos\theta + y^{\cdot} \sin\theta)(\alpha\xi - \beta\varphi - \gamma\chi) = 0$$

In the absence of deformations $(\xi = 0, \varphi = 0,$ and $\gamma = 0)$ the second of Eqs. (3) becomes the identity $\theta^{\cdot} = kv$.

We denote the mass and weight of the wheel by m and $N = mg$; the diametral and axial moments of inertia by A and B; the coordinates of the wheel center C in the coordinate system O_1xyz by x_C, y_C, z_C; the wheel radius by r, and the angle of the wheel proper rotation by ϑ. The kinetic potential is defined by formula

$$L = \frac{m}{2}(x_C^{\cdot 2} + y_C^{\cdot 2} + z_C^{\cdot 2}) + \frac{A}{2}(\chi^{\cdot 2} + \theta^{\cdot 2}\cos^2\chi) + \frac{B}{2}(\vartheta^{\cdot} + \theta^{\cdot}\sin\chi)^2 - Nz_C, \quad z_C = r\cos\chi$$

Assuming, as in [1,3,4], that the variables ξ, φ, χ are small, we obtain the equation of motion of the form

$$m\,(U^{\cdot} + V\theta^{\cdot} - r\chi^{\cdot\cdot} + r\theta^{\cdot 2}\chi) + a\xi + \sigma N\chi = 0 \tag{4}$$

$$A\theta^{\cdot\cdot} + B\omega\chi^{\cdot} - b\varphi = 0$$

$$A\chi^{\cdot\cdot} + [N(\rho + \sigma r - r) - (B - A)\,\theta^{\cdot 2}]\chi - B\omega\theta^{\cdot} + (ar + \sigma N)\,\xi = 0$$

$$U - V\varphi - \xi^{\cdot} = 0, \quad \theta^{\cdot} + \varphi^{\cdot} - k\,(V + \xi\theta^{\cdot}) - V(\alpha\xi - \beta\varphi - \gamma\chi) = 0$$

$$(F = a\xi + \sigma N\chi, \quad M_\theta = b\varphi, \quad M_\chi = -\sigma N\xi - \rho N\chi)$$

where $V = \omega r = \text{const}$ represents the variable v in the case of uniform motion, and the constant quantities a, b, σ, ρ are proportional to partial derivatives of the lateral force F, moments M_θ and M_χ with respect to corresponding deformations [1].

Since $\xi \ll R$, hence from the last equation of system (4) we have

$$\theta^{\cdot} = \Omega - \varphi^{\cdot} + V(\alpha + R^{-2})\,\xi - \beta V\varphi - \gamma V\chi, \quad \Omega = V / R \tag{5}$$

The remaining equations of this system are of the form

$$\xi^{\cdot\cdot} + [am^{-1} + V^2(\alpha + R^{-2})]\,\xi - r\chi^{\cdot\cdot} + (\sigma g + r\Omega^2 - \gamma V^2)\,\chi - \beta V^2\varphi = -V\Omega \tag{6}$$

$$\varphi^{\cdot\cdot} + \beta V\varphi^{\cdot} + bA^{-1}\varphi + (\gamma V - B\omega A^{-1})\,\chi^{\cdot} - V(\alpha + R^{-2})\,\xi^{\cdot} + 2VR^{\cdot}R^{-3}\xi = \Omega^{\cdot}$$

$$A\chi^{\cdot\cdot} + [B\gamma\omega V + N(\rho + \sigma r - r) - (B - A)\Omega^2]\chi + B\omega\varphi^{\cdot} + B\beta\omega V\varphi + [ar + \sigma N - B\omega V(\alpha + R^{-2})]\,\xi = B\omega\Omega$$

$$U = \xi^{\cdot} + V\varphi$$

Let us consider the particular solution of system (6), (5) when point K moves along a circle of radius R

$$U_0 = 0,\ \varphi_0 = 0,\ \theta_0^{\cdot} = \Omega + V(\alpha + R^{-2})\,\xi_0 - \gamma V \chi_0 = \text{const}$$

$$\xi_0 = \Delta_1 / \Delta = \text{const},\ \chi_0 = \Delta_2 / \Delta = \text{const}$$

$$\Delta_1 = mV\Omega\,[A\Omega^2 + B\sigma g r^{-1} - N(r - r\sigma - \rho)],\quad \Delta_2 = -V\Omega\,[aBr^{-1} + m(ar + \sigma N)]$$

$$\Delta = -Am\Omega^4 + \Omega^2\,[a(B - A + mr^2) + mN(r - \rho) - B\sigma N r^{-1} - Am\alpha V^2] + V^2\,[mN\alpha(r - r\sigma - \rho) - B\alpha\sigma N r^{-1} - m\gamma(ar + \sigma N) - Ba\gamma r^{-1}] + N\,[\sigma^2 N + a(r - \rho)]$$

We call this equation unperturbed. Note that $\Delta_2 < 0$ always. The remaining quantities are considered for the following numerical data:

$m = 98.1$ kg, $r = 0.5$ m, $A = 6.131$ kgm^2, $B = 2A$, $a = 98100$ kgs^{-2},

$b = 3924$kgm^{2}s^{-2}, $\alpha = 20$m^{-2}, $\beta = 10$ m^{-1}, $\gamma = 1$ m^{-1}, $\rho = 0.1$m, $\sigma = 0.6$

We vary parameters V and R. Computations show that $\Delta_1 > 0$. Hence $\xi_0\Delta > 0$ and $\chi_0\Delta < 0$.

The intersection of surface $\Delta = \Delta(V, R)$ with the plane R = const, i.e. the line $\Delta = \Delta_R(V)$ of the one-parameter set of curves with R as the parameter is shown in Fig. 2. When $V \in (0, V_*)$ we have $\xi_0 > 0$ and $\chi_0 < 0$, and when $V > V_*$ then $\xi_0 < 0$ and $\chi_0 > 0$. The dependence of ξ_0 and χ_0 on R for $V = 5$, 15, and 50 m/s is shown in Fig. 3 by curves *1*, *2*, and *3*, respectively. Function $V = V_*(R'/r)$ is shown in Fig. 4 by the solid line. As parameter R is increased, V_* first rapidly decreases and, then, becomes virtually stable.

To test the stability of the unperturbed solution with respect to variables $U, \theta^{\cdot}, \xi, \varphi$ and χ we consider the variational equations that correspond to (5) and (6). We seek a solution of the system of variational equations in the form of products of a constant factor by $\exp(pt)$ and obtain the following characteristic equation:

$$p^6 + a_1 p^5 + a_2 p^4 + a_3 p^3 + a_4 p^2 + a_5 p + a_6 = 0$$

$$a_1 = \beta V,\quad a_2 = a_{31} + bA^{-1} + \alpha V^2 + \Omega^2,\quad a_{31} = V^2 b_1^2 r^{-2} + B_1 + (1 - b_1)\,\Omega^2$$

$$b_1 = BA^{-1},\quad B_1 = am^{-1} + B_2 r - mN_1,\quad B_2 = (ar + \sigma N)A^{-1},$$

$$N_1 = g(r - r\sigma - \rho)A^{-1}$$

$$a_3 = a_1 a_{31},\quad a_4 = bB_1A^{-1} - B_3 + V^4\alpha b_1^2 r^{-2} + V^2\,[bb_1\gamma A^{-1} r^{-1} + B_2\gamma + ab_1^2 m^{-1} r^{-2} + \alpha b(1 - b_1)A^{-1} - \alpha m N_1] + V^2\Omega^2\,[b_1^2 r^{-2} + \alpha(1 - b_1)] - \Omega^2\,[B_5 + 2(b_1 - 1)\,bA^{-1} + mN_1] + (1 - b_1)\,\Omega^4,\quad B_3 = B_2\sigma g + aN_1$$

$$B_4 = B_2 + ab_1 m^{-1} r^{-1},\quad B_5 = B_2 r + a(b_1 - 1)\,m^{-1},\quad N_2 = b_1\sigma g r^{-1} - mN_1$$

$$a_5 = a_1\,(V^2 B_4 b_1 r^{-1} - \Omega^2 B_5 - B_3),\quad a_6 = -bmA^{-2}\Delta$$

Numerical analysis shows that the unperturbed motion is stable when the single condition $a_6 > 0$, i.e. when $\Delta < 0$. Hence the mode $V < V_*, \xi_0 > 0$, and $\chi_0 < 0$ is unstable, and mode $V > V_*, \xi_0 < 0, \chi_0 > 0$ stable. In the stable motion mode the wheel with a pneumatic tire tilts away from the circle center owing to the predominant effect of the centrifugal force moment $mV^2 r / R$.

In the same formulation but using the hypothesis of outward bias (instead of that of Keldysh) in an unperturbed motion for the variables we have

$$\theta_0^{\cdot} = \Omega = \frac{V}{R}, \quad U_0 = \frac{mV^2\Omega\,(A\Omega^2 - Nr)}{a_1\,[Nr + (B - A + mr^2)\,\Omega^2]},$$
$$\chi_0 = -\frac{\Omega V\,(B + mr^2)}{r\,[Nr + (B - A + mr^2)\,\Omega^2]}$$

where $a_1 = \mathrm{const} > 0$ is the coefficient of outward bias, always $B > A$, $\chi_0 < 0$; U_0 is an alternating-sign quantity. A manifold of circular motions of the wheel is stable when conditions

$$V > V_0, \quad V > V_1$$
$$V_0^2 = ANr^3\,[B\,(B + mr^2) - A\,(B - A + mr^2)r^2 / R^2]^{-1}$$
$$V_1^2 = ANr^3\,[B\,(B - A) + A\,(2A - B)r^2 / R^2]^{-1}$$

are simultaneously satisfied. In the case of a disk we have the single condition $V^2 > 2gr$ and then

$$\chi_0 = {}^{-3}/_2 V^2\,(Rg + {}^5/_4 V^2 r / R)$$

In the case of an absolutely rigid wheel moving along a circle with observance of classic nonholonomic relationships of rolling, the small tilt angle is determined by the same formula as in the case of application of the outward bias hypothesis, and the condition of conservative stability of the manifold of circular motions is of the form $V > V_0$.

For the disk

$$V_0^2 = gr\left[3\left(1 - \frac{5}{12}\frac{r^2}{R^2}\right)\right]^{-1}$$

which for $R \to \infty$ yields the known conditions $V^2 > gr / 3$ of rectilinear motion.

Let us compare the stability of the wheel circular motion for various models of interaction between the wheel periphery and the road surface. In all three cases the wheel can move along the circle at any speed. The stability region is bounded in the RV plane on the left by the straight line $R = r$ and from below by the curve $V = V_*(R)$. The lower stability boundaries are shown in Fig. 4, where the solid curve relates to the elastic wheel (in conformity with Keldysh's theory) and the dash lines *1* and *2* relate to the outward bias hypothesis for a rigid wheel.

According to the outward bias hypothesis $V_*(R) = \mathrm{const}$ the largest stability region for a specific wheel obtains for the rigid wheel, when the curve $V = V_*(R)$ has a horizontal asymptote $V = (gr / 3)^{1/2}$. When $R / r > 6$, function $V_*(R)$ decreases so slowly for both the elastic and rigid models that it can be considered as virtually constant. The difference in the stability regions of the three indicated models is only quantitative.

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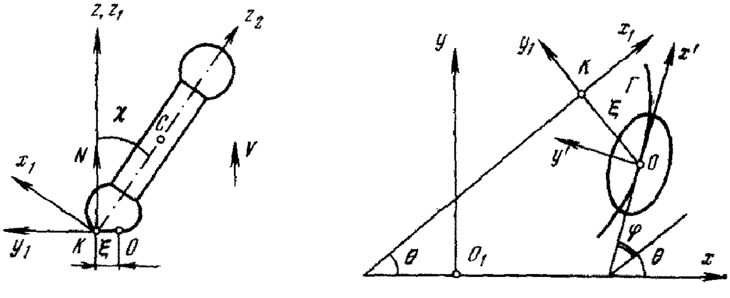

Fig. 1

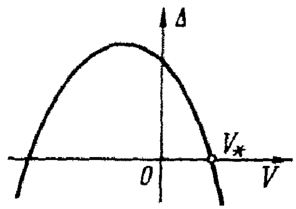

Fig. 2

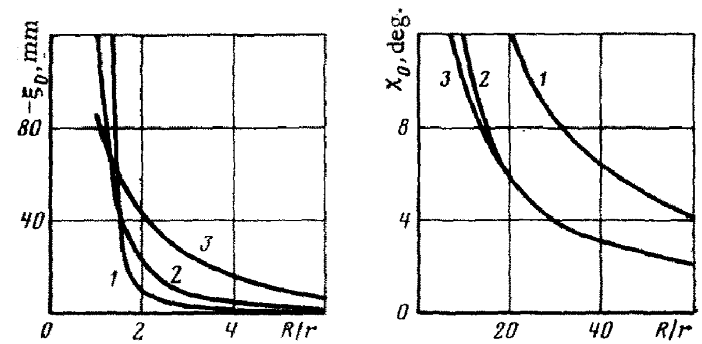

Fig. 3

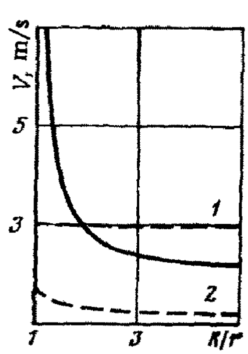

Fig. 4

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